Applied Regression Modeling: A Business Approach
Chapter 5: Regression Model Building II
Sections 5.1–5.2

by Iain Pardoe

Influential points

- Beware model conclusions that are overly influenced by a small handful of data points, e.g.:
  - overall results can be biased if a few unusual points differ dramatically from general patterns in the majority of the data values;
  - misleading to conclude evidence of a strong association between variables if evidence based mainly on a few dominant points.

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- Focus on two measures of individual data point influence:
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  - *outliers* have unusual Y-values relative to their predicted $\hat{Y}$-values from a model;
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- *Cook's distance* is a composite measure of outlyingness and leverage.

Outliers

- Outliers have unusual Y-values relative to their predicted $\hat{Y}$-values from a model.
- In other words, observations with a large magnitude residual, $\hat{e}_i = Y_i - \hat{Y}_i$.
- Computer can calculate *studentized* residuals to put them on a common scale.
- When four regression assumptions (zero mean, constant variance, normality, and independence) are satisfied, studentized residuals $\approx \mathrm{N}(0, 1^2)$.
- If we identify an observation with a studentized residual outside $(-3, 3)$, we've either witnessed a very unusual event (one with prob. less than 0.002) or we've found an observation with a Y-value that doesn't fit the pattern in the rest of the dataset.
- Formally define a potential outlier as an observation with studentized residual < -3 or > 3.

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  - potential outliers differ substantively from other sample observations (remedy: remove outliers and reanalyze remainder of dataset separately).
- To gauge outlier influence exclude largest magnitude studentized residual, refit model to remaining observations, and see if regression parameter estimates change substantially.

CARS5 data

- Y = city miles per gallon (MPG) for 50 new U.S. passenger cars in 2004.
- X_1 = weight (thousands of pounds).
- X_3 = engine size (liters).
- X_5 = wheelbase (hundreds of inches).
- Model:
  $\mathrm{E}(Y) = b_0 + b_1(1/X_1) + b_2(1/X_3) + b_3(1/X_5)$.

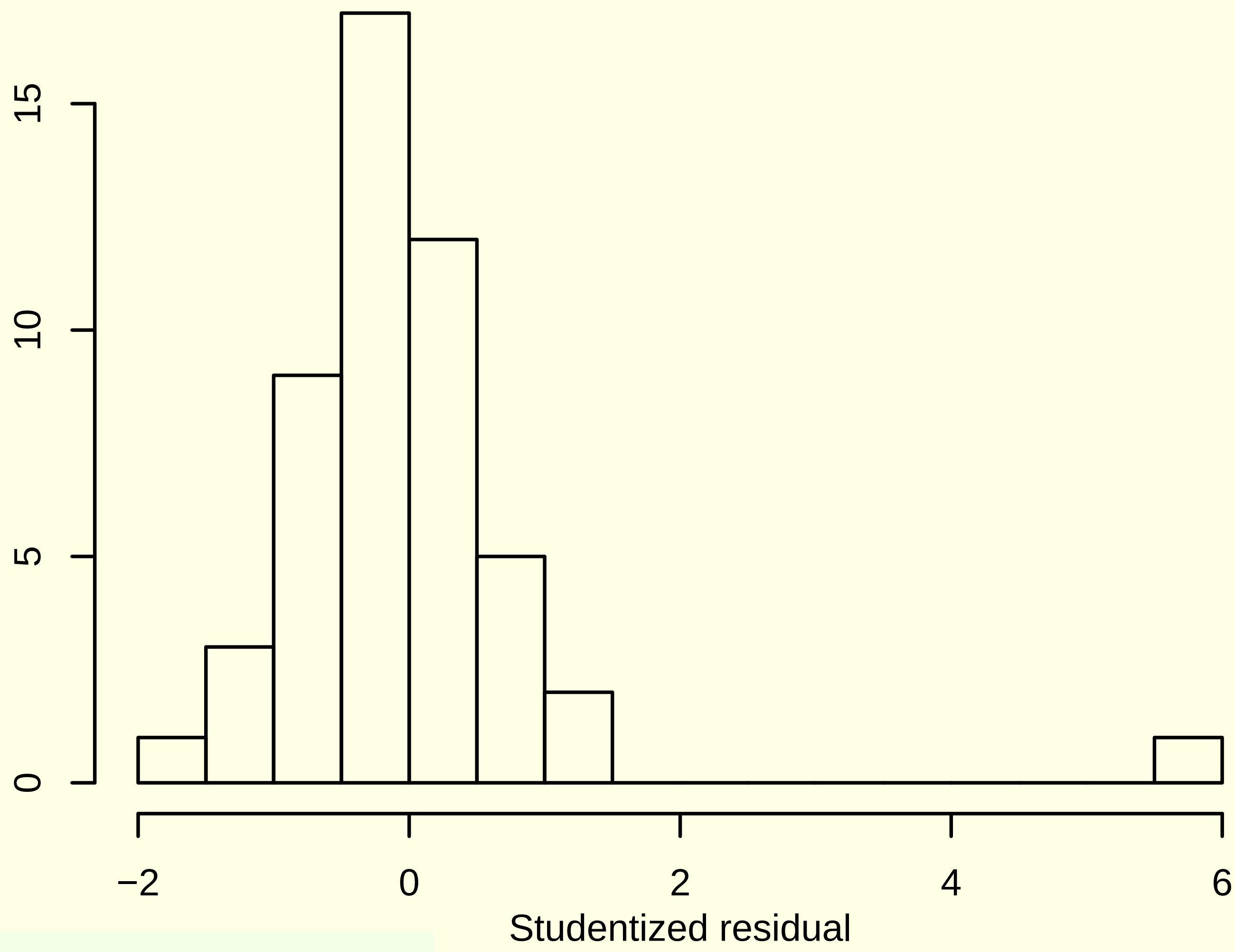
Parameters [a]

Model		Estimate	Std. Error	t-stat	Pr(> |t|)
1	(Intercept)	9.397	13.184	0.713	0.480
	recipX1	44.296	13.173	3.363	0.002
	recipX3	19.404	6.706	2.894	0.006
	recipX5	−9.303	17.087	−0.544	0.589

[a] Response variable: Y.

Model 1 studentized residuals

There is one outlier with studentized residual ≈ 6.



Remove outlier

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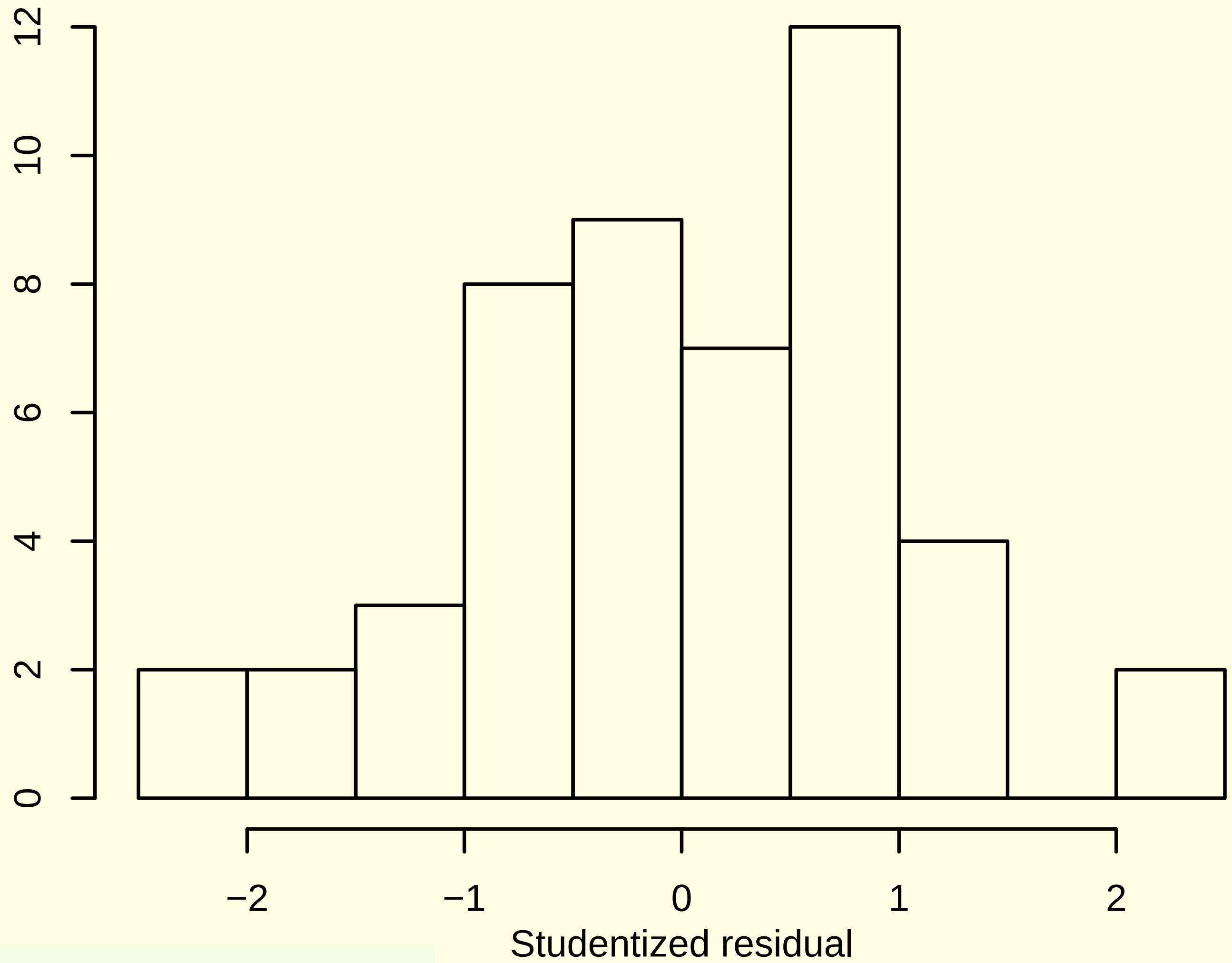
Model		Estimate	Std. Error	t-stat	Pr(> |t|)
2	(Intercept)	25.946	7.612	3.409	0.001
	recipX1	64.071	7.682	8.340	0.000
	recipX3	17.825	3.782	4.713	0.000
	recipX5	−33.106	9.919	−3.338	0.002

[a] Response variable: Y.

- Regression parameter estimates and p-values change dramatically.
- The outlier was diesel-powered and did not fit the pattern of the gasoline-powered cars.

Model 2 studentized residuals

No further outliers.



Leverage

- High *leverage* points have unusual combinations of X-values relative to general dataset patterns.
- If a point is far from the majority of the sample, it can pull the fitted model close toward its Y-value, potentially biasing the results.
- Leverage measures potential for an observation to have undue influence on a model (0–1: low–high).

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- To gauge influence exclude largest leverage point, refit model to remaining observations, and see if reg. parameter estimates change substantially.

Results with outlier removed

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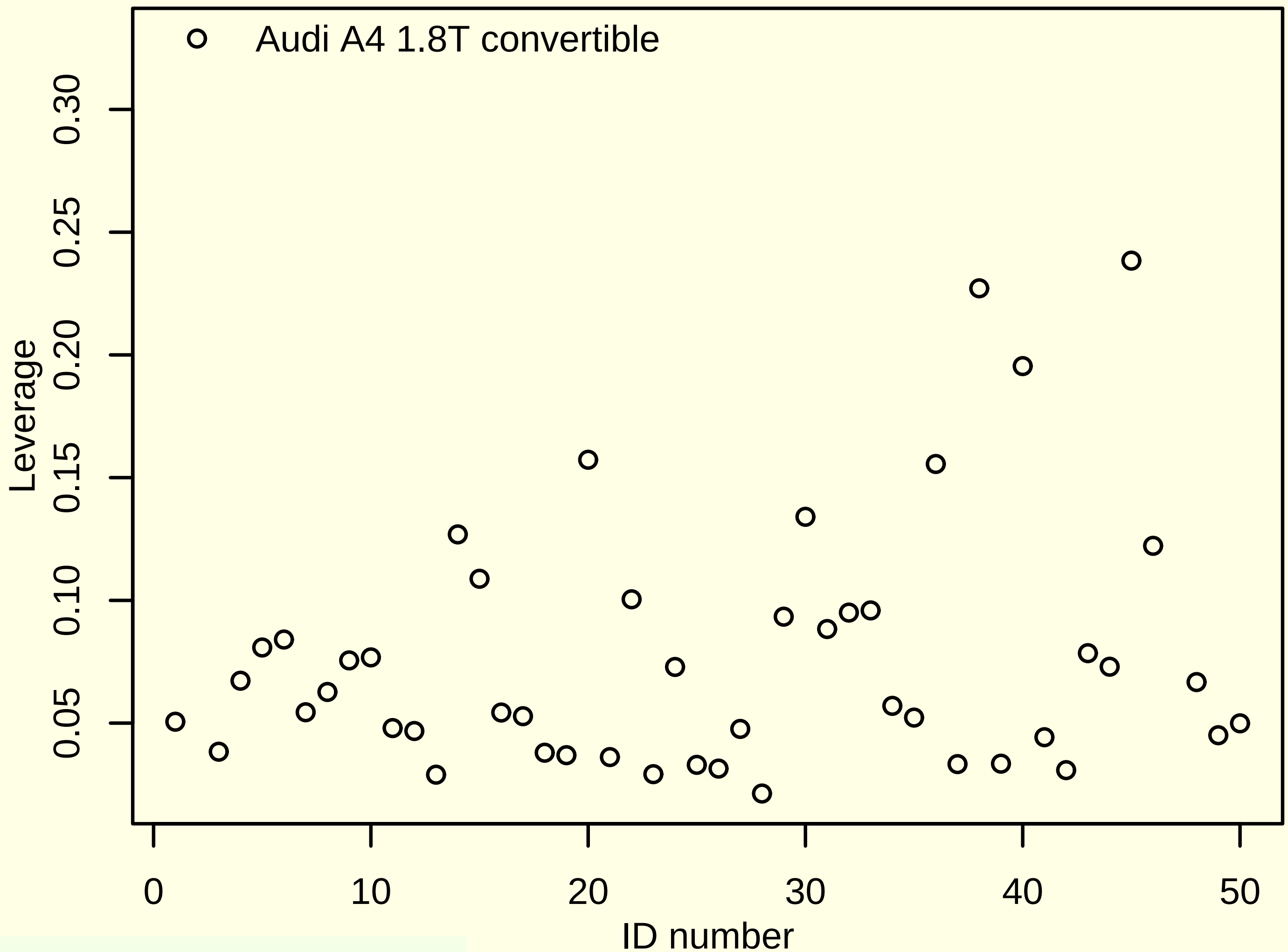
[a] Response variable: Y.

- Threshold: $3(k+1)/n = 3(3+1)/49 = 0.24.$
- Threshold: $2(k+1)/n = 2(3+1)/49 = 0.16.$

Model 2 leverages

Highest leverage point exceeds $3(k+1)/n$ threshold.



Results with high leverage point removed

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Model		Estimate	Std. Error	t-stat	Pr(>|t|)
3	(Intercept)	24.811	7.717	3.215	0.002
	recipX1	68.054	8.785	7.747	0.000
	recipX3	15.743	4.389	3.587	0.001
	recipX5	−32.400	9.960	−3.253	0.002

[a] Response variable: Y.

- Regression parameter estimates and p-values don't change dramatically.
- The high leverage point had the potential to strongly influence results, but in this case did not do so.

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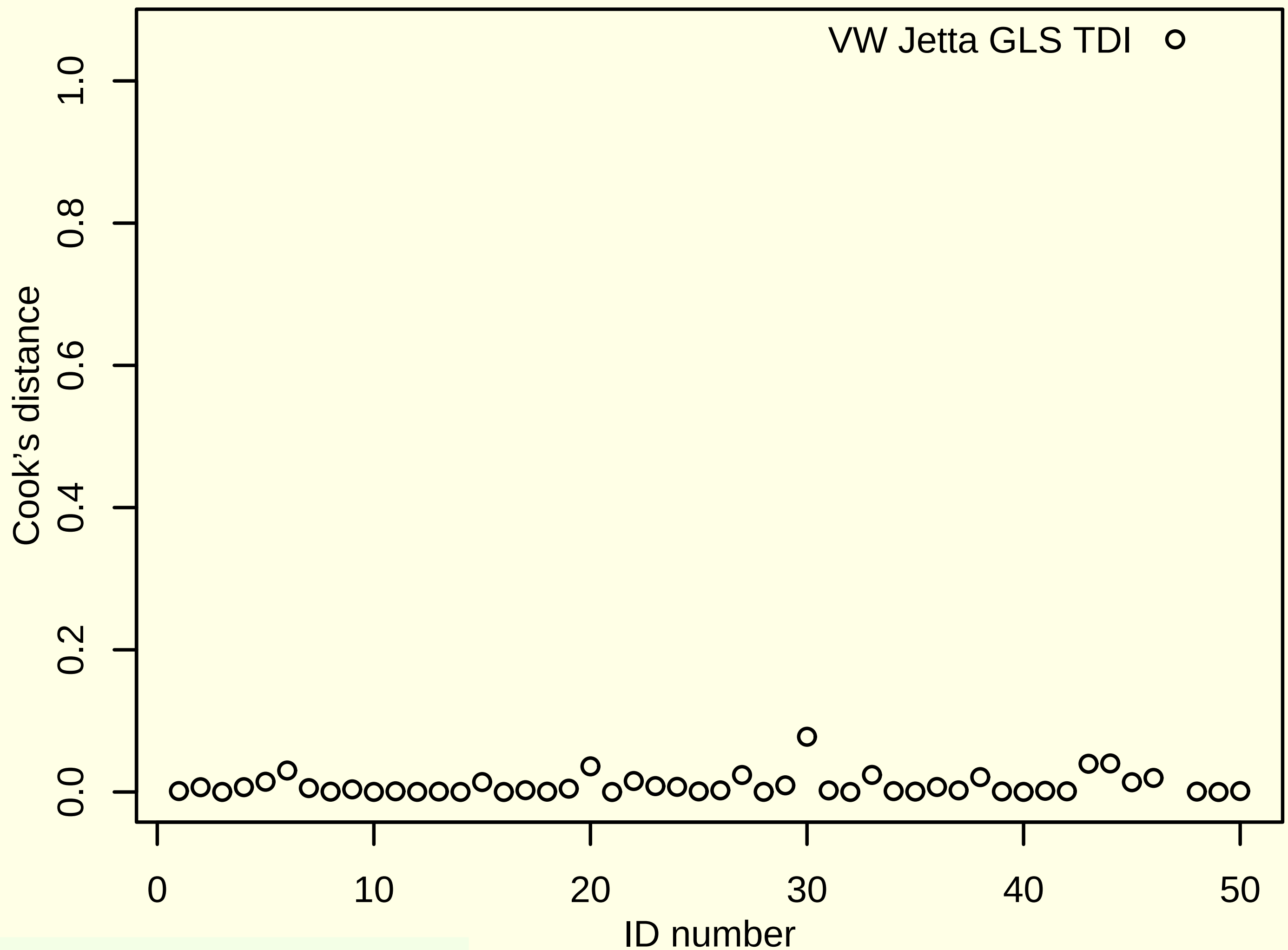
[a] Response variable: Y.

Model 1 Cook's distances

Highest Cook's distance exceeds 1 threshold.



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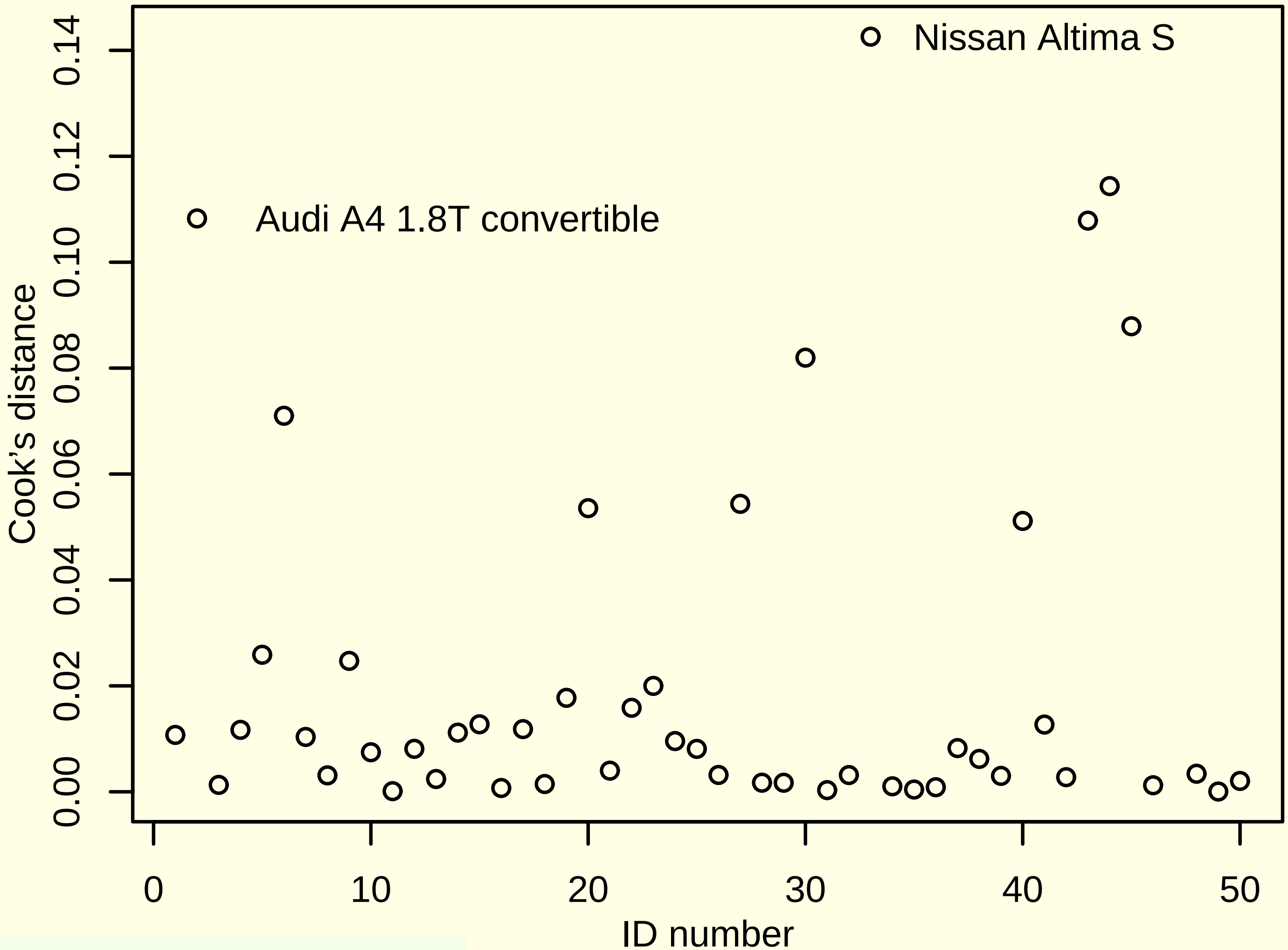
[a] Response variable: Y.

- The car with the highest Cook's distance was the outlier we found before.

Model 2 Cook's distances

Highest Cook's distance less than 0.5 threshold.



- Some of the pitfalls that can cause problems with a regression analysis:
  - autocorrelation (serial correlation)—failing to account for time trends in the model;
  - multicollinearity—highly correlated predictors causing unstable model results;
  - excluding important predictor variables—leading to possibly incorrect conclusions;
  - overfitting (the sample data)—leading to poor generalizability to the population;
  - extrapolation—using model results for predictor values very different to those in the sample;
  - missing data—leading to reduced sample sizes at best, misleading results at worst.

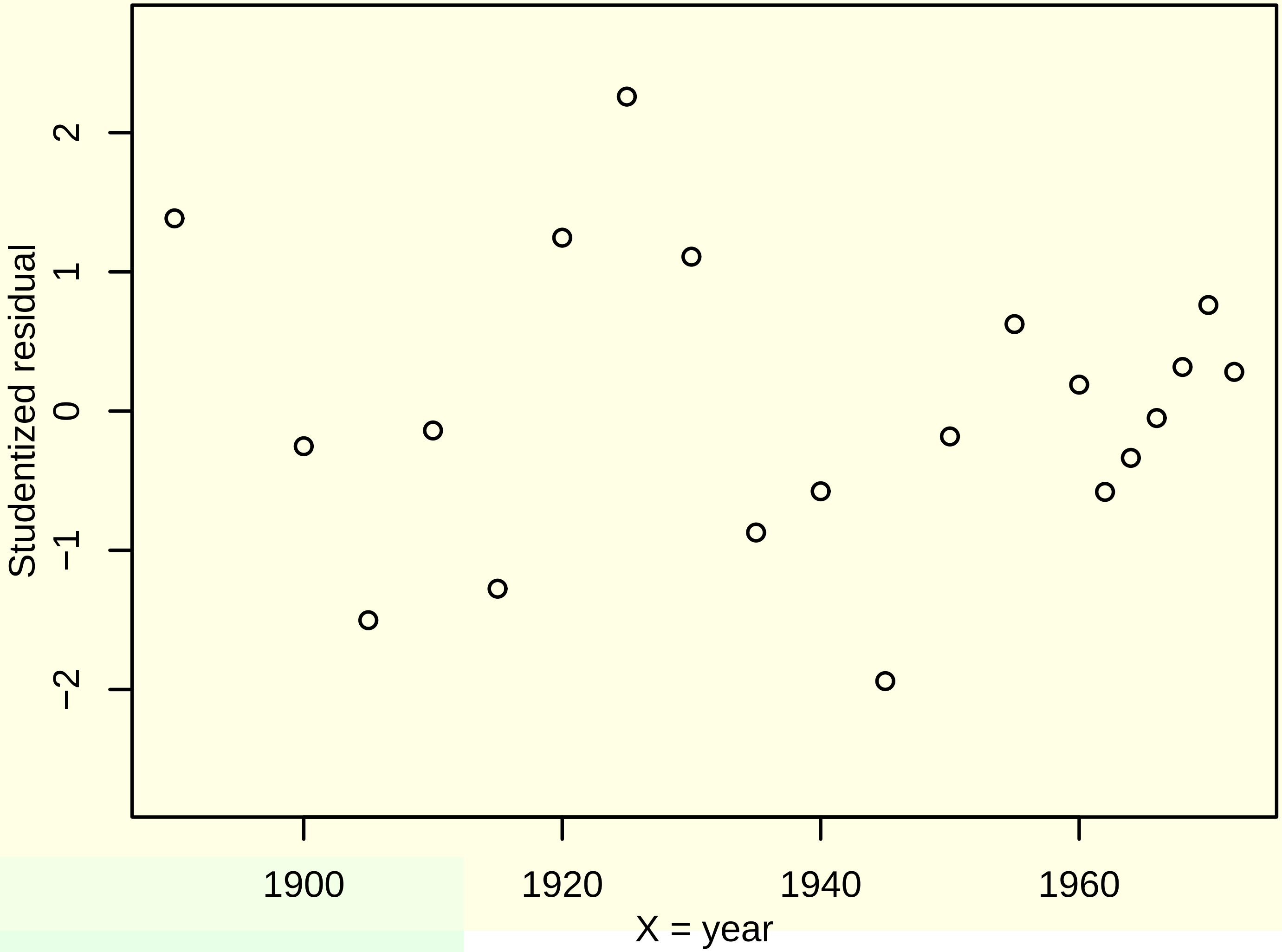
Autocorrelation

- Autocorrelation occurs when regression model residuals violate the independence assumption because they are highly dependent across time.
- Can occur when regression data have been collected over time and model fails to account for any strong time trends.
- Dealing with this issue rigorously can require specialized time series and forecasting methods.
- Sometimes, however, simple ideas can mitigate autocorrelation problems.
- Example: **OIL** data file contains annual world crude oil production in millions of barrels (Y) from 1880 to 1972 (X).
- Model 1: $\mathrm{E}(\log_e(Y)) = b_0 + b_1 X$.

Plot shows clear evidence of autocorrelation.

Model 2 residuals

Model 2: $\mathrm{E}(\log_e(Y_t)) = b_0 + b_1 X_t + b_2 \log_e(Y_{t-1})$.
Independent errors assumption more reasonable now.



Multicollinearity

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- Example: **SALES3** data file with sales (Y), TV/newspaper advertising (X_1), and internet advertising (X_2).

Model 1 results

- Model 1: $\mathrm{E}(Y) = b_0 + b_1 X_1 + b_2 X_2$.

Model Summary

Model	Multiple R	R Squared	Adjusted R Squared	Regression Std. Error
1	0.987 [a]	0.974	0.968	0.8916

[a] Predictors: (Intercept), X1, X2.

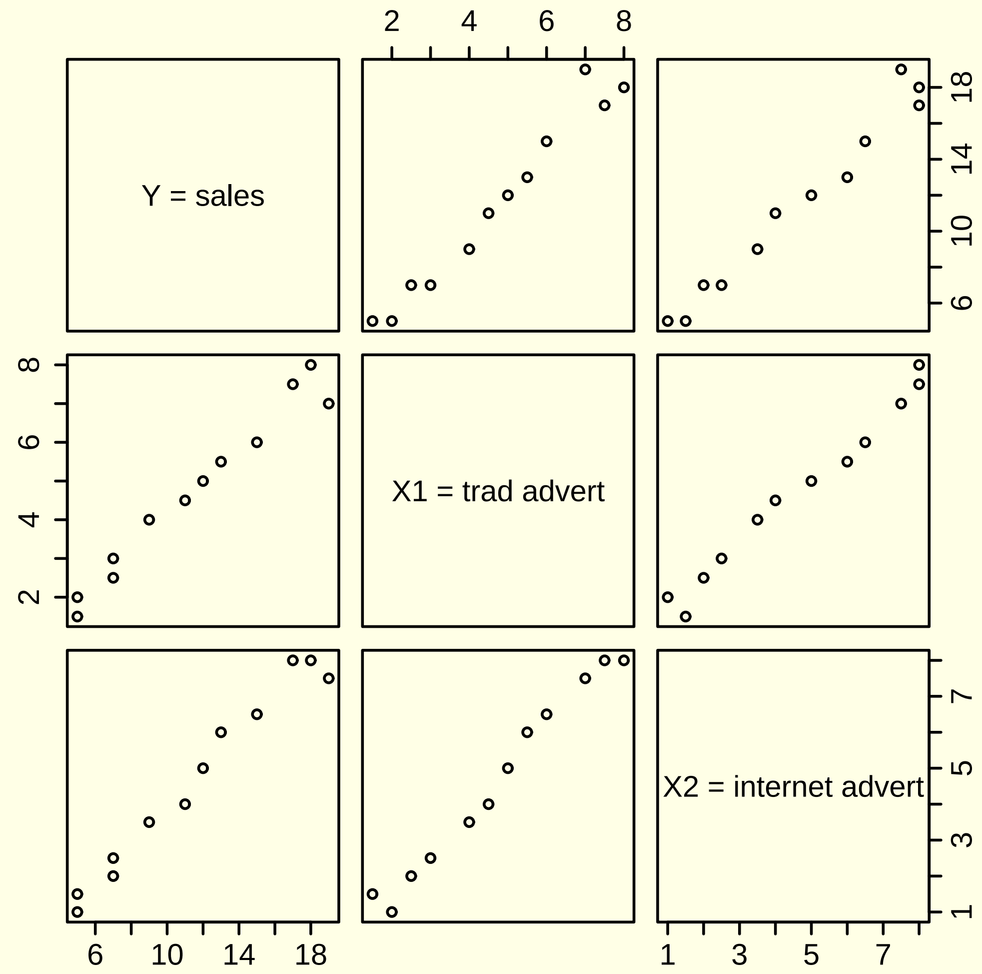
Parameters [a]

Model		Estimate	Std. Error	t-stat	Pr(>|t|)	VIF
1	(Intercept)	1.992	0.902	2.210	0.054	
	X1	0.767	0.868	0.884	0.400	49.541
	X2	1.275	0.737	1.730	0.118	49.541

[a] Response variable: Y.

- R^2 is 0.974, but neither X_1 nor X_2 are significant (given the presence of the other)!
- VIF > 10 suggests there is a multicollinearity problem.

Unstable estimates when both X_1 and X_2 in model.



Model 2 results

- Model 2: $\mathrm{E}(Y) = b_0 + b_1(X_1 + X_2)$.

Model Summary

Model	Multiple R	R Squared	Adjusted R Squared	Regression Std. Error
2	0.987 [a]	0.974	0.971	0.8505

[a] Predictors: (Intercept), X1plusX2.

Parameters [a]

Model		Estimate	Std. Error	t-stat	Pr(>|t|)
2	(Intercept)	1.776	0.562	3.160	0.010
	X1plusX2	1.042	0.054	19.240	0.000

[a] Response variable: Y.

- R^2 unchanged, and the combined predictor variable, $X_1 + X_2$, is significant.
- Note this approach is only possible if it makes sense to create a combined predictor variable.
- More common to drop one of the correlated predictors from model.

Excluding important predictor variables

- Excluding important predictors sometimes results in models that provide incorrect, biased conclusions about included predictors.
- Strive to include all potentially important predictors, and remove a predictor only if there are compelling reasons to do so (e.g., if causing multicollinearity problems and has high individual p-value).
- Example: **PARADOX** data file with $n = 27$ high-precision computer components with component quality (Y) potentially depending on two controllable machine factors, speed (X_1) and angle (X_2).

Model 1 results

- Model 1: $\mathrm{E}(Y) = b_0 + b_1 X_1$.

Parameters [a]

Model		Estimate	Std. Error	t-stat	Pr(> |t|)
1	(Intercept)	2.847	1.011	2.817	0.009
	X1	0.430	0.188	2.288	0.031

[a] Response variable: Y.

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- In other words, increase the speed of the machine to improve quality.

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- Results suggest a positive association between quality and speed.
- In other words, increase the speed of the machine to improve quality.
- However, this ignores process information relating to angle.

Model 2 results

- Model 2: $\mathrm{E}(Y) = b_0 + b_1 X_1 + b_2 X_2$.

Parameters [a]

Model		Estimate	Std. Error	t-stat	Pr(>|t|)
2	(Intercept)	1.638	0.217	7.551	0.000
	X1	−0.962	0.071	−13.539	0.000
	X2	2.014	0.086	23.473	0.000

[a] Response variable: Y.

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- Results suggest a *negative* association between quality and speed (for a fixed angle),
- and a positive association between quality and angle (for a fixed speed).

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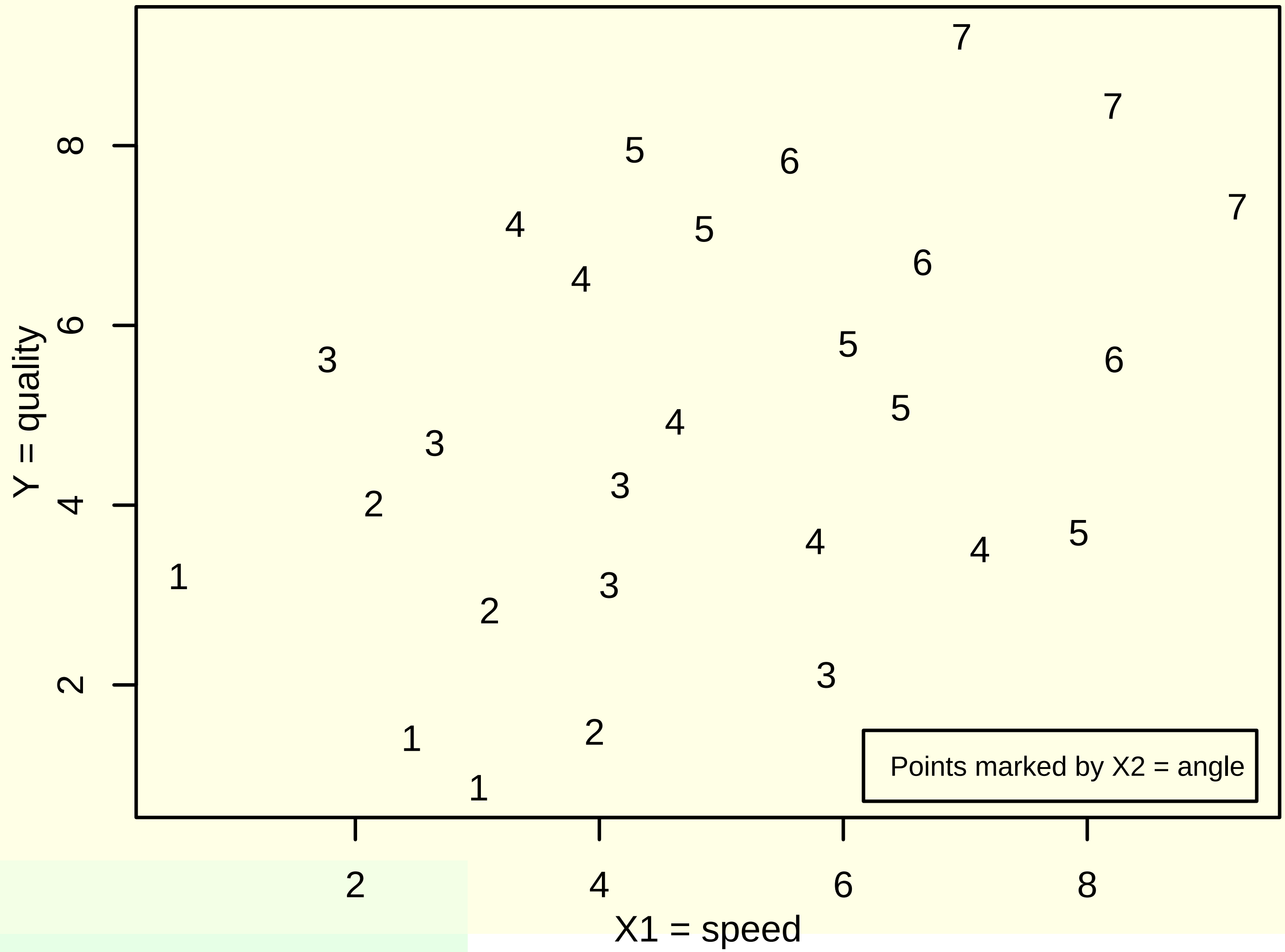
- and a positive association between quality and angle (for a fixed speed).
- In other words, increase the angle but decrease the speed of the machine to improve quality.

Paradox explained

Positive association between Y and X_1 ignoring X_2.
Negative association between Y and X_1 accounting for X_2.



Overfitting

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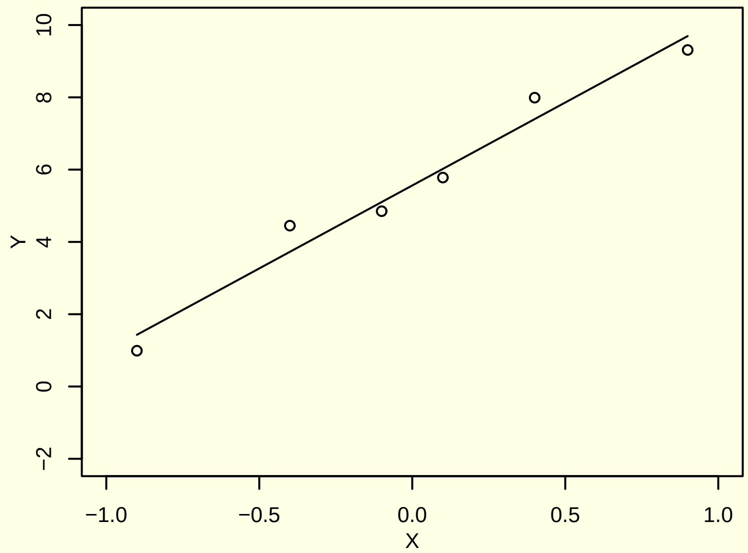
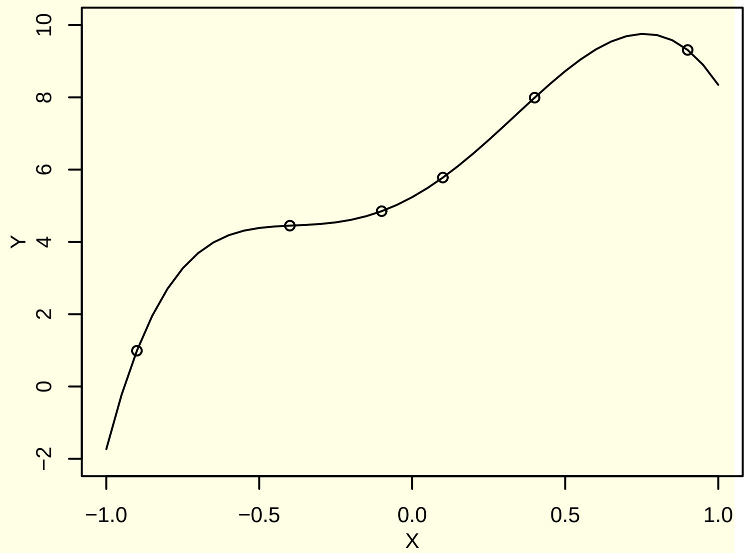




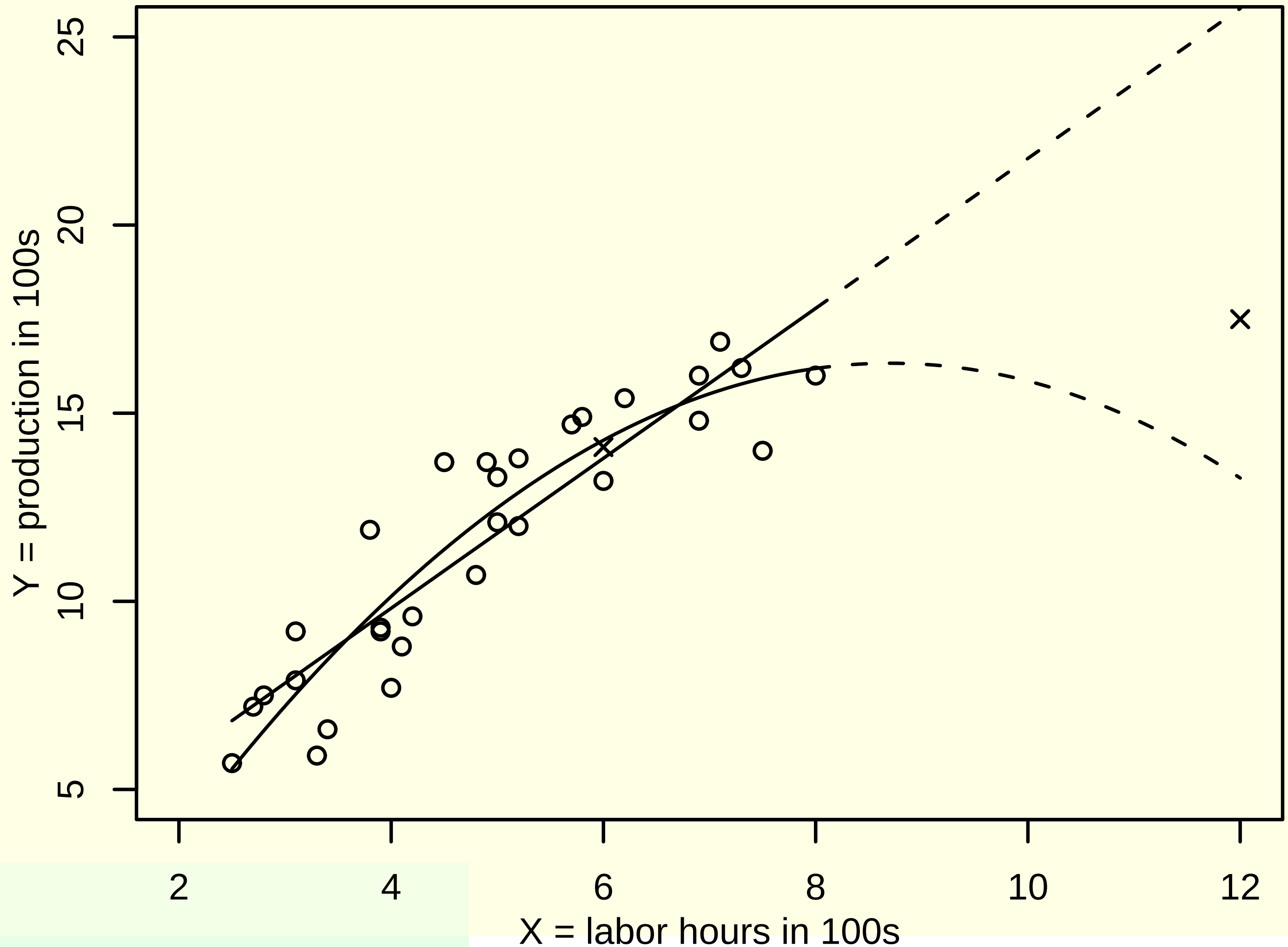
- Which model seems more reasonable?

Extrapolation

- Extrapolation occurs when regression model results are used to estimate or predict a response value for an observation with predictor values that are very different from those in the sample.
- This can be dangerous because it means making a decision about a situation where there are no data values to support our conclusions.
- Example: if we observe an upward trend between two variables, should we assume the trend continues indefinitely at higher values?

Extrapolation example

Straight-line model overshoots actual Y at far right. Quadratic model undershoots (i.e., neither model enables accurate prediction far from sample data).



Missing data

- Missing data occurs when particular values in the dataset have not been recorded for particular variables and observations.
- Dealing with issue rigorously is beyond scope of book, but there are some simple ideas that can mitigate some of the major problems.
- Example: **MISSING** data file with $n = 30$, Y, and X_1–X_4.
- No missing values for Y, X_1, or X_4, but five missing values for X_2, one of which is also missing X_3:

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  - including X_3 (but excluding X_2) will exclude one observation;
  - excluding X_2 and X_3 will exclude no observations.

Predictors	Sample size	R^2	s
X_1, X_2, X_3, X_4	25	0.959	0.865
X_2, X_3, X_4	25	0.958	0.849
X_1, X_3, X_4	29	0.953	0.852
X_1, X_4	30	0.640	2.300

- Ordinarily, we would probably favor the (X_2, X_3, X_4) model.
- However, the (X_1, X_3, X_4) model applies to much more of the sample.
- Thus, in this case, we would probably favor the (X_1, X_3, X_4) model, since R^2 and s are roughly equivalent, but the usable sample size is much larger.